

Longitudinal conductivity of a three-dimensional Dirac electron gas in magnetic field

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Abstract. We discuss a conductivity of a three-dimensional gas of Dirac electrons in the direction of magnetic field in high magnetic fields. Analytical expressions for the zero temperature conductivity are calculated in the linear response theory using the basis of relativistic Landau levels. The impurity scattering is treated in the self-consistent Born approximation which gives Landau level broadening increasing linearly in magnetic field. We demonstrate that in the special case of zero-gap Dirac semimetal this leads to the magnetic field and temperature independent conductivity in high magnetic field limit.

1. Introduction

Over the last year there has been a considerable progress in experimental investigations of transport properties in three-dimensional (3D) Dirac electron systems [1, 2, 3, 4]. A prominent real material where the existence of a zero gap 3D Dirac electron phase was predicted in the first principle calculations [5] and then confirmed experimentally [6] is Cd_3As_2 . The electrical resistivity measurements in this material demonstrated that Cd_3As_2 had a giant positive magnetoresistance in the direction perpendicular to the magnetic field which is strongly sample dependent. In particular, the low-mobility samples showed linear magnetoresistance, while in the high-mobility samples the resistivity exhibited nearly quadratic magnetic-field dependence in the quantized magnetic field region where Shubnikov–de Haas oscillations were found [1, 2]. Although the theory of quantum linear transverse magnetoresistance was developed by Abrikosov [7], its origin in Cd_3As_2 [1, 2, 3], TlBiSSe [4] and other 3D Dirac semimetals is not completely understood at the present time. While the transverse magnetoresistance demonstrated a giant dependence on the magnetic field, the longitudinal resistivity in Cd_3As_2 in the same situation became nearly temperature independent in magnetic fields larger than 2 T and showed pronounced Shubnikov–de Haas oscillations.

In the present paper we provide a theoretical discussion of the longitudinal conductivity in 3D Dirac electron system. The case of transverse and Hall conductivities will be considered in the subsequent publication. From the theoretical point of view, the magnetoconductivity of the zero-gap 3D Weyl and Dirac semimetals was considered recently by Gorbar et al. [8]. In particular case of the longitudinal conductivity, the authors showed that in high magnetic fields Weyl and Dirac semimetals possessed the same conductivity, which contained “topological contribution” from the $n = 0$ Landau level which is independent of the temperature and the



chemical potential and proportional to B/Γ_0 where B is a magnetic field and Γ_0 is a broadening of the zero Landau level .

The present paper is devoted to the longitudinal conductivity of Dirac materials with a finite gap Δ in the energy spectrum. For the zero-gap case we confirm the result obtained in Ref. [8]. However, in the present paper we calculate the Landau level broadening Γ in the self-consistent Born approximation (SCBA) which gives a magnetic field dependence of Γ . Similar to the two-dimensional case, where in SCBA $\Gamma \sim \sqrt{B}$ in the quantum limit [9], we found that in the 3D case Γ increases linearly in magnetic field. In the small-gap case the magnetic field dependence of Γ leads to the magnetic field (and temperature) independent conductivity in high magnetic fields.

The paper is organised as follows. In section 2 we develop a formalism. Some useful analytical expressions for the conductivity are also derived in Appendix A. In section 3 we discuss the magnetic-field and chemical potential dependences of the Landau level broadening and the conductivity. Section 4 is reserved for the summary.

2. Formalism

2.1. Energy spectrum and wave functions

We consider a relativistic tree-dimensional electron gas in external magnetic field with the Hamiltonian

$$\hat{H} = \begin{pmatrix} \Delta & v\boldsymbol{\sigma} \cdot (\mathbf{p} + e\mathbf{A}) \\ v\boldsymbol{\sigma} \cdot (\mathbf{p} + e\mathbf{A}) & -\Delta \end{pmatrix}, \quad (1)$$

where v is the Fermi velocity, Δ is a gap in an energy spectrum, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ denotes the vector of the Pauli matrices, $\mathbf{p} = -i\hbar\nabla$ is a momentum operator with $\hbar = h/2\pi$ being the Plank's constant, an electron charge is $-e$ ($e > 0$), and \mathbf{A} is a magnetic vector potential. The vector potential is chosen in the Landau gauge $\mathbf{A} = (-By, 0, 0)$ which implies that the magnetic field \mathbf{B} is applied along the z -axis.

In the Landau gauge the solution of an eigenvalue problem for the Hamiltonian (1) leads to relativistic Landau levels

$$E_n^{(\eta)} = \eta\sqrt{\epsilon_c^2 n + \epsilon_z^2 + \Delta^2}, \quad n = 0, 1, 2, \dots \quad (2)$$

where $\eta = \pm 1$ is a band index, $\epsilon_c = \sqrt{2eB\hbar v^2}$ is a cyclotron energy, and $\epsilon_z = vp_z = v\hbar k_z$. We imply that the system has a translation invariance along x - and z -directions where periodical boundary conditions are imposed, and k_x and k_z denote the corresponding wave numbers. The eigenfunctions have the following form [10, 11]

$$\psi_{ns}^{(\eta)}(\mathbf{r}) = \frac{e^{ik_x x + ik_z z}}{\sqrt{\ell L_x L_z}} \begin{pmatrix} \alpha_n^{(\eta)} A_n^{(s)} \phi_{n-1}(\xi) \\ -s\alpha_n^{(\eta)} A_n^{(-s)} \phi_n(\xi) \\ \eta s \alpha_n^{(-\eta)} A_n^{(s)} \phi_{n-1}(\xi) \\ -\eta \alpha_n^{(-\eta)} A_n^{(-s)} \phi_n(\xi) \end{pmatrix}, \quad (3)$$

where

$$\alpha_n^{(\eta)} = \left[\frac{1}{2} \left(1 + \eta \frac{\Delta}{E_n} \right) \right]^{\frac{1}{2}}, \quad (4)$$

and

$$A_n^{(s)} = \left[\frac{1}{2} \left(1 + \frac{s\epsilon_z}{\sqrt{E_n^2 - \Delta^2}} \right) \right]^{\frac{1}{2}}. \quad (5)$$

Hereafter, E_n denotes $|E_n^{(\eta)}|$. In equation (3), $\ell = \sqrt{\hbar/(eB)}$ is magnetic length, L_j ($j = x, y, z$) is the size of the system in the j -direction, $\xi = y/\ell + \ell k_x$, and ϕ_n is a wave function of a harmonic oscillator

$$\phi_n(\xi) = \frac{e^{-\frac{\xi^2}{2}}}{2^{\frac{n}{2}} \pi^{\frac{1}{4}} \sqrt{n!}} H_n(\xi), \quad (6)$$

with H_n being the n th Hermite polynomial. Each energy level is multiply degenerated with respect to k_x with the degeneracy factor $L_x L_y / (2\pi \ell^2)$, and each level with $n > 0$ is doubly degenerated with respect to $s = \pm 1$. The zero mode ($n = 0$ eigenstate) is obtained from equations (3), (4), and (5) if we set $s = -\epsilon_z/|\epsilon_z|$, $A_0^{(s)} = 0$, and $A_0^{(-s)} = 1$. The explicit form of the zero mode is

$$\psi_0^{(\eta)}(\mathbf{r}) = \frac{e^{ik_x x + ik_z z}}{\sqrt{\ell L_x L_z}} \phi_0(\xi) \begin{pmatrix} 0 \\ \alpha_0^{(\eta)} \\ 0 \\ -\frac{\epsilon_z}{|\epsilon_z|} \eta \alpha_0^{(-\eta)} \end{pmatrix}, \quad E_0^{(\eta)} = \eta \sqrt{\epsilon_z^2 + \Delta^2}. \quad (7)$$

Therefore, for $n > 0$ each quantum state is labelled by n, k_x, k_z, η, s , while for $n = 0$ state the quantum numbers are k_x, k_z , and η . Note that the variables defined in equations (1), (4), and (5) have k_z -dependence which is not indicated for brevity.

2.2. Longitudinal electrical conductivity

The diagonal part of the electrical conductivity of an electron gas in a magnetic field can be expressed in the following form [12, 13]

$$\sigma_{jj}(T, \mu) = -\frac{e^2 \hbar}{\pi} \int_{-\infty}^{\infty} d\varepsilon \frac{\partial f(\varepsilon)}{\partial \varepsilon} \text{Tr} \left[\hat{v}_j \text{Im} \hat{G}(\varepsilon) \hat{v}_j \text{Im} \hat{G}(\varepsilon) \right], \quad j = x, y, z, \quad (8)$$

where $f(\varepsilon) = \{1 + \exp[(\varepsilon - \mu)/T]\}^{-1}$ is the Fermi-Dirac distribution function with chemical potential μ and temperature T , the Boltzmann constant $k_B = 1$, and $\text{Im} \hat{G}(\varepsilon) = \frac{i}{2} [\hat{G}^{(+)}(\varepsilon) - \hat{G}^{(-)}(\varepsilon)]$ where

$$\hat{G}^{(\pm)}(\varepsilon) = (\varepsilon - \hat{\mathcal{H}} \pm i\delta)^{-1}, \quad \delta \rightarrow 0^+, \quad (9)$$

with $\hat{\mathcal{H}}$ being the total Hamiltonian which includes the scattering over impurities. An explicit form of $\hat{\mathcal{H}}$ will be specified later, however, hereafter we imply that the effect of impurities is included into the electrons self-energy $\hat{\Sigma}(\varepsilon)$ since vertex corrections vanish for the conductivity of an electron gas in a magnetic field with short-ranged scatterers [12]. The velocity operator in this case is defined as $\hat{\mathbf{v}} = \frac{i}{\hbar} [\hat{H}, \mathbf{r}]$ which for Dirac electrons has the following explicit form

$$\hat{\mathbf{v}} = v \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}. \quad (10)$$

In the present paper we discuss the conductivity along the direction of magnetic field, i. e. $\sigma_{zz}(T, \mu)$, which will be referred to as $\sigma_{\parallel}(T, \mu)$. If the trace in equation (8) is calculated with respect to the wave function (2), σ_{\parallel} at $T = 0$ becomes

$$\sigma_{\parallel}(0, \mu) = \frac{e^2 \hbar}{2\pi^2 \ell^2} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \sum_{nn'} \sum_{\eta\eta'} \sum_{ss'} |\langle n'\eta's' | \hat{v}_z | n\eta s \rangle|^2 \mathcal{A}_{n'}^{(\eta')}(\mu) \mathcal{A}_n^{(\eta)}(\mu), \quad (11)$$

where we have already taken into account that matrix elements of $\hat{\mathbf{v}}$ are diagonal in k_z , and have no k_x dependence. The summation over k_z is changed to the integration and after the the

limits of the integration are expanded to the infinity. The Green's functions in the r.h.s. are determined as

$$\mathcal{A}_n^{(\eta)}(\mu) = \frac{\Gamma_n^{(\eta)}(\mu)}{(\mu - E_n^{(\eta)} - \mathcal{R}_n^{(\eta)}(\mu))^2 + \Gamma_n^{(\eta)2}(\mu)} \quad (12)$$

where $\mathcal{R}_n^{(\eta)}$ and $\Gamma_n^{(\eta)}$ refer to the real and imaginary part of the self-energy $\Sigma_n^{(\eta)}$ respectively. For \hat{v}_z , an explicit form of the matrix elements is

$$\langle n' \eta' s' | \hat{v}_z | n \eta s \rangle = v \delta_{nn'} \left(s \eta \alpha_n^{(-\eta)} \alpha_{n'}^{(\eta')} + s' \eta' \alpha_{n'}^{(-\eta')} \alpha_n^{(\eta)} \right) \left[A_{n'}^{(s')} A_n^{(s)} - s' s A_{n'}^{(-s')} A_n^{(-s)} \right]. \quad (13)$$

As far as there is no dependence on s or s' in equation (12), we can make a summation over s and s' in equation (11), taking into account that for the $n = 0$ we should make a substitution $A_0^{(s')} = A_0^{(s)} = 0$, $A_0^{(-s')} = A_0^{(-s)} = 1$, and remove the summation over s and s' . After algebraically cumbersome, but straight forward calculations, we obtain

$$\sum_{ss'} |\langle n' \eta' s' | \hat{v}_z | n \eta s \rangle|^2 = v^2 \alpha_n \left(\frac{1}{2} - \frac{1}{2} \eta \eta' + \eta \eta' \frac{\epsilon_z^2}{E_n^2} \right), \quad \alpha_n = 2 - \delta_{n0}. \quad (14)$$

With the help of equation (14), we arrive to the following expression for the zero temperature conductivity

$$\sigma_{\parallel}(0, \mu) = \frac{e^2 v^2}{4\pi^3 \ell^2} \int_{-\infty}^{\infty} dp_z \sum_{n=0}^{\infty} \sum_{\eta} \alpha_n \left[\frac{\epsilon_z^2}{E_n^2} \mathcal{A}_n^{(\eta)2} + \left(1 - \frac{\epsilon_z^2}{E_n^2} \right) \mathcal{A}_n^{(\eta)} \mathcal{A}_n^{(-\eta)} \right]. \quad (15)$$

As we discuss below, in the case when the self-energy in equation (12) depends only on μ , the summation over n or the integration over p_z in equation (15) can be performed analytically which produces expressions suitable for numerical calculations. The details can be found in Appendix A.

2.3. Self-consistent Born approximation for impurity scattering

We consider the total Hamiltonian in equation (9) in the following form

$$\hat{\mathcal{H}} = \hat{H} + u \sum_l \delta(\mathbf{r} - \mathbf{R}_l), \quad (16)$$

where we imply that the point like impurities are distributed randomly at the positions \mathbf{R}_l with the concentration n_s and single-impurity potential u . As was demonstrated by Bastin et al. [12], for point-like impurities the self-energy in the self-consistent Born approximation is determined from the equation

$$\Sigma(\varepsilon) = \frac{n_s u^2}{V} \text{Tr} \hat{G}(\varepsilon), \quad (17)$$

where V is a volume of the system. Since \hat{G} is diagonal in the Landau level basis in equation (2), the trace is calculated immediately, and an expanded form of equation (17) reads

$$\Sigma(\varepsilon) = \frac{n_s u^2}{2\pi \ell^2} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \sum_{n=0}^{\infty} \sum_{\eta} \frac{\alpha_n}{\varepsilon - E_n^{(\eta)} - \Sigma(\varepsilon)}, \quad \alpha_n = 2 - \delta_{n0}, \quad (18)$$

The self-energy, therefore, depends only on ε . In order to have better understanding of equation (18), we consider the following particular cases.

2.3.1. Clean limit approximation. In the clean limit approximation we imply that the imaginary part of the self energy is negligible $\Gamma(\varepsilon) \rightarrow 0^+$, and replace r.h.s. of equation (18) by a δ -function. In this case, the self-consistency is lost and we are in situation of the first Born approximation. Integrating over k_z we obtain

$$\Gamma(\varepsilon) = \frac{n_s u^2}{v h \ell^2} \sum_{n=0}^{\infty} \alpha_n \frac{|\varepsilon| \theta(|\varepsilon| - \Delta)}{\sqrt{\varepsilon^2 - \Delta^2 - \epsilon_c^2 n}}. \quad (19)$$

In the equation (19), $\Gamma(\varepsilon)$ as a function of ε has a characteristic saw-tooth profile, however, in the clean limit the r.h.s. contains divergences at $\varepsilon \approx E_n$ which are cured in a self-consistent case.

2.3.2. High magnetic field approximation and ζ -regularization. In order to obtain an expression for the self-energy suitable for the analyses in high magnetic fields, we make a summation over η in equation (18) and integrate over k_z which gives a self-consistent equation

$$\Sigma = -\frac{n_s u^2}{v h \ell^2} \sum_{n=0}^{\infty} \alpha_n \frac{\varepsilon - \Sigma}{\sqrt{\epsilon_c^2 n + \Delta^2 - (\varepsilon - \Sigma)^2}}. \quad (20)$$

This expression is not restricted to the high magnetic field case, however, the r.h.s contains a divergent series which behaves as $n^{-\frac{1}{2}}$, and proper regularization is needed. In the case of quantizing magnetic fields, when energy levels are well separated, we can keep in the summation only the term with $\varepsilon \sim E_n^{(\eta)}$, and the divergence becomes irrelevant.

In the general case, to treat the divergent series of this type, one can try ζ -regularization, which is frequently used in the high-energy physics [14]. In the ζ -regularization technique, the series in the r.h.s. of equation (20) is expressed via the Hurwitz ζ -function $\zeta(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s}$ which can be continued analytically for $s < 1$. By using analytical continuation, the uniquely defined finite value is assigned to the divergent series. With the help of this technique, we get the following answer

$$\Sigma = \frac{n_s u^2 \epsilon_c^2}{4\pi v^3 \hbar^3} (\varepsilon - \Sigma) \left[\frac{1}{\sqrt{\Delta^2 - (\varepsilon - \Sigma)^2}} - \frac{2}{\epsilon_c} \zeta \left(\frac{1}{2}, \frac{\Delta^2 - (\varepsilon - \Sigma)^2}{\epsilon_c^2} \right) \right], \quad (21)$$

Although equation (21) can be useful for practical calculations, the physical meaning of this regularization in the present case is obscure for the authors.

In order to have some insight on the meaning of ζ -regularization, it is instructive to consider asymptotic behaviour of equation (21) at $\epsilon_c \rightarrow \infty$. The Hurwitz ζ -function has the following asymptotic series [14]

$$\zeta(z+1, a) = \frac{1}{z} a^{-z} + \frac{1}{2} a^{-z-1} + \frac{1}{z} \sum_{k=2}^{\infty} \frac{B_k}{k!} \frac{\Gamma(z+k)}{\Gamma(z)} a^{-z-k}, \quad (22)$$

where B_k are Bernoulli numbers and $\Gamma(z)$ is the Euler's gamma-function. Using this expansion, equation (21) in the $\epsilon_c \rightarrow \infty$ limit reads

$$\Sigma = \frac{n_s u^2}{\pi v^3 \hbar^3} (\varepsilon - \Sigma) \sqrt{\Delta^2 - (\varepsilon - \Sigma)^2}. \quad (23)$$

This expression has to be compared with the self-energy calculated at zero magnetic field directly from the Hamiltonian in equation (1).

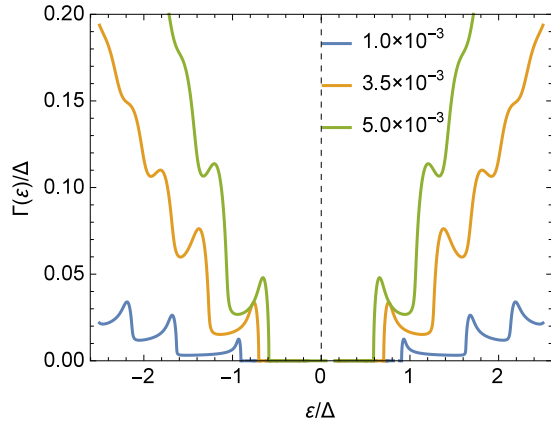


Figure 1. (color online) $\Gamma(\varepsilon)$ as a function of ε for $\kappa = 0.001, 0.0035$, and 0.005 with $\epsilon_c/\Delta = 1.5$ calculated from equation (20) with 150 Landau levels.

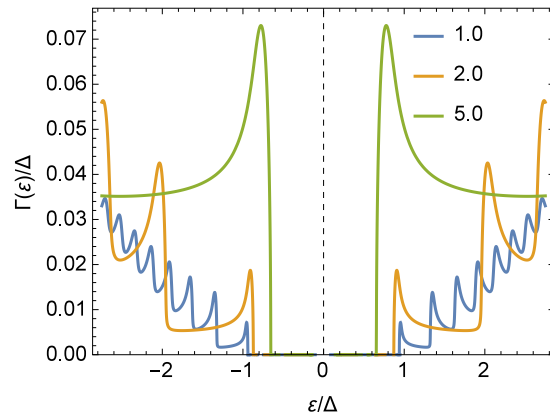


Figure 2. (color online) $\Gamma(\varepsilon)$ as a function of ε with fixed $\kappa = 0.001$ for $\epsilon_c/\Delta = 1, 2$, and 5 calculated from equation (20) with 150 Landau levels.

2.3.3. Zero magnetic field case. In the zero magnetic field the energy spectrum of the Dirac equation consists of two bands $E_{\mathbf{p}}^{(\eta)} = \eta\sqrt{\Delta^2 + v^2 p^2}$, each band is doubly degenerated with respect to the two possible projection of electron spin on the direction of \mathbf{p} . The self-consistent equation (17) in this case gives

$$\Sigma = \frac{2n_s u^2}{\hbar^3} \int_{\text{BZ}} \frac{d\mathbf{p}}{(2\pi)^3} \sum_{\eta} \frac{1}{\varepsilon - E_{\mathbf{p}}^{(\eta)} - \Sigma}, \quad (24)$$

where the integration is over the Brillouin zone. In the clean limit approximation, where $\Gamma(\varepsilon) \rightarrow 0^+$, we obtain the following result

$$\Gamma(\varepsilon) = \frac{n_s u^2}{\pi \hbar^3 v^3} |\varepsilon| \sqrt{\varepsilon^2 - \Delta^2} \theta(|\varepsilon| - \Delta). \quad (25)$$

In the general case, if we consider free electrons, the r.h.s in equation (24) diverges linearly

$$\Sigma = -2(\varepsilon - \Sigma) \frac{n_s u^2}{\pi^2 \hbar^3} \int_0^\infty dp \frac{p^2}{v^2 p^2 + \Delta^2 - (\varepsilon - \Sigma)^2} \quad (26)$$

Using the cut-off parameter $\frac{\pi}{2}\Lambda$ for the integral, we obtain the self-consistent equation

$$\Sigma = -\frac{n_s u^2}{\pi \hbar^3 v^3} (\varepsilon - \Sigma) \left[\Lambda - \sqrt{\Delta^2 - (\varepsilon - \Sigma)^2} \right]. \quad (27)$$

which differs from equation (23) by a presence of cut-off parameter. For a practical usage of equation (21), we can add manually the term $-4\Lambda/\epsilon_c^2$ to the brackets in the r.h.s of (21) to have consistence with equation (27).

3. Results and discussion

3.1. Landau level broadening in magnetic field

We start the discussion with analysing the behaviour of the Landau level broadening in the magnetic field. In figure 1, $\Gamma(\varepsilon)$ is shown as a function of ε/Δ for several values of dimensionless

parameter $\varkappa = \Delta/\epsilon_s$ where $\epsilon_s = 4\pi v^3 \hbar^3 / (n_s u^2)$ is a characteristic energy of the scattering. At small \varkappa , Γ as a function of ε has a characteristic for a three-dimensional system saw-tooth profile in agreement with the clean limit approximation in equation (19) where in the self-consistent approximation the divergences at $\epsilon = E_n$ become regularized. With increasing \varkappa , the Landau levels become broader and the gap between $E_0^{(-)}$ and $E_0^{(+)}$ energy levels reduces leading, finally, to the disappearance of the Landau quantization and the closing of the gap for large impurity concentration.

Energy dependence of the $\Gamma(\varepsilon)$ for several ϵ_c is demonstrated in figure 2. With increasing ϵ_c both level separation and level broadening increase. In order to understand how $\Gamma(\varepsilon)$ depends on the magnetic field with fixed ε , let us consider equation (20) in the clean limit $\epsilon_c/\epsilon_s \ll 1$ when Landau levels are well separated. In high magnetic fields we can keep only the term with $n = 0$ in the r.h.s.

$$\Sigma_0 = -\frac{n_s u^2 \epsilon_c^2}{4\pi v^3 \hbar^3} \frac{\varepsilon - \Sigma_0}{\sqrt{\Delta^2 - (\varepsilon - \Sigma_0)^2}}. \quad (28)$$

In the case of negligible Δ , we can neglect Δ^2 with respect to $(\varepsilon - \Sigma_0)^2$, which gives $\Gamma_0(\varepsilon)$ independent from ε for $|\varepsilon| < \epsilon_c$ and proportional to the magnetic field

$$\Gamma_0(\varepsilon) = \frac{\epsilon_c^2}{\epsilon_s} = \frac{n_s u^2 e B}{2\pi \hbar^2 v}. \quad (29)$$

Figure 3 illustrates $\Gamma(\varepsilon)$ in the zero-gap case for several values of ϵ_c/ϵ_s which shows flat $\Gamma(\varepsilon)$ in the vicinity of $\varepsilon = 0$. From equation (29) we find that with increasing ϵ_c , the Landau level broadening increases faster than the level separation which leads to the violation of the clean limit approximation when $\epsilon_c \approx \epsilon_s$ where the magnetic field dependence of Γ_0 changes.

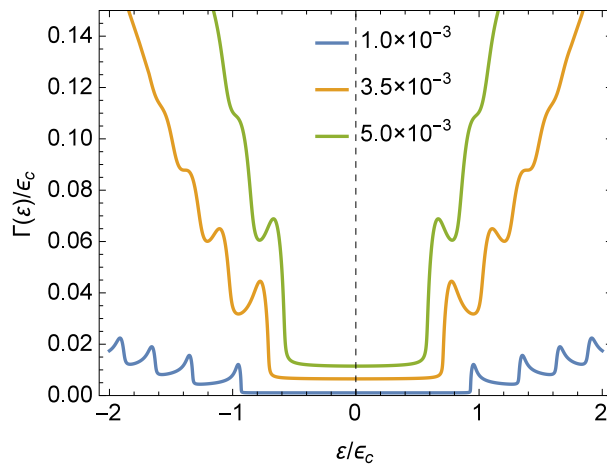


Figure 3. (color online) $\Gamma(\varepsilon)$ as a function of ε/ϵ_c for several ϵ_c/ϵ_s with $\Delta = 0$. In the vicinity of $\varepsilon = 0$, $\Gamma(\varepsilon)$ becomes independent of ε and given by equation (29).

3.2. Conductivity as a function of chemical potential

First, we would like to comment on the difference between the constant Γ approximation, which is frequently used to analyse transport properties, and the self-consistent Born approximation applied in the present paper. Figure 4 shows conductivity as a function of μ/ϵ_c for $\Delta/\epsilon_c = 0, 0.25$, and 1 calculated with constant $\Gamma/\epsilon_c = 0.01$ from equation (A.7) using numerical summation over 150 Landau levels. An important point is that $\sigma_{||}$ with $\Gamma = \text{const}$ increases monotonously with increasing $|\mu|$. In the zero gap case $\sigma_{||}$ becomes flat in the segment $|\mu| < \epsilon_c$ where it is given by equation (A.8). This result coincide with a longitudinal conductivity for a Weyl semimetal

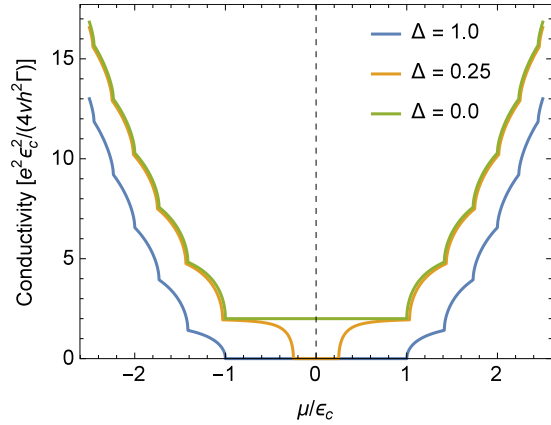


Figure 4. (color online) $\sigma_{\parallel}(0, \mu)$ as a function of μ for several Δ in the constant- Γ approximation with $\Gamma/\epsilon_c = 0.01$.

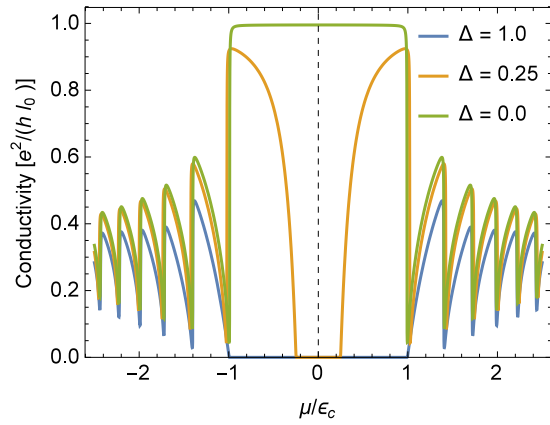


Figure 5. (color online) $\sigma_{\parallel}(0, \mu)$ as a function of μ for several Δ/ϵ_c in the self-consistent Born approximation.

obtained by Gorbar et al. [8]. The conductivity at $\mu = 0$ in this case increases linearly in magnetic field and independent of temperature for $\epsilon_c > k_B T$.

However, if we use self-consistent Born approximation for $\Gamma(\epsilon)$ the result changes considerably. Figure 5 shows the $\sigma_{\parallel}(0, \mu)$ with the self-energy calculated from equation (19) for $\epsilon_c/\epsilon_s = 10^{-4}$. In the self-consistent Born approximation σ_{\parallel} as a function of μ shows pronounced oscillations due to oscillating behaviour of $\Gamma(\epsilon)$ (see figure 2). In particular, in the zero gap case $\Gamma(\epsilon) \sim B$ and conductivity for $|\mu| < \epsilon_c$ becomes independent of magnetic field and chemical potential

$$\sigma_0 = \frac{e^2}{h} \frac{1}{\ell_0}, \quad (30)$$

where $\ell_0 = 2n_s u^2 / (\hbar^2 v^2)$ is some characteristic length scale which is proportional to the impurity concentration.

Finally, we show the conductivity as a function of a chemical potential for several ϵ_c with finite gap and $\kappa = 10^{-4}$. Similar, to the zero-gap case, the conductivity for $\mu \approx \Delta$ is independent of the magnetic field for fixed μ . However, in real materials with finite doping the chemical potential depends on magnetic field since electron concentration is fixed and the conductivity with finite gap becomes magnetic field and temperature dependent.

4. Summary

In summary, we have considered the longitudinal conductivity of a 3D gas of Dirac fermions in the magnetic field in the framework of the Kubo linear response theory. The scattering on the impurities was treated using the self-consistent Born approximation. The application of the self-consistent Born approximation results in the linear magnetic field dependence of the Landau level broadening in high magnetic fields and clean system. Using this level broadening we calculated the chemical potential dependence of the conductivity. In the zero gap case our result for the conductivity is in agreement with the result obtained previously by Gorbar et al. for the case of a 3D Weyl semimetal [8]. However, in our case linear magnetic-field dependence of the level broadening leads to the magnetic field independent conductivity at $\mu = 0$ in high magnetic fields where the contribution from the $n = 0$ Landau level becomes dominant.

Finally, we comment on the application of our results to the recent experiments. The longitudinal transport measurements in zero-gap 3D Dirac electron systems demonstrated that the magnetoresistance in the direction of the magnetic field was negligible with respect to the

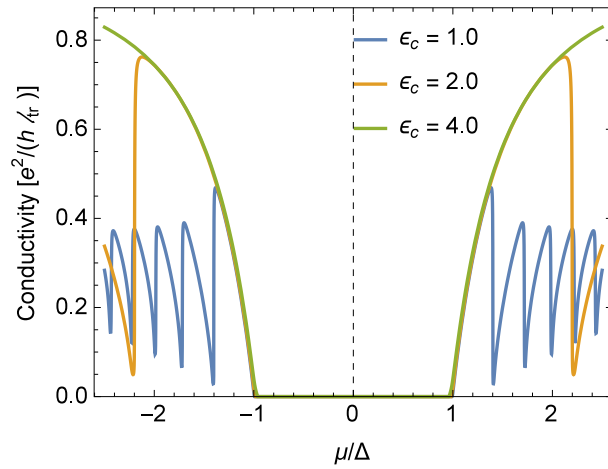


Figure 6. (color online) Conductivity as a function of μ for several ϵ_c with $\kappa = 10^{-4}$ calculated from equations (A.7) and (20) with 150 energy levels.

giant transverse magnetoresistance [1, 2, 3, 4]. This is in agreement with our result that the zero-gap conductivity becomes independent of the magnetic field in quantum limit. However, in Cd_3As_2 negative temperature independent magnetoresistance was reported in the magnetic fields greater than 2 T for the low mobility sample [1]. This might indicate that in Cd_3As_2 magnetic-field dependence of the level broadening is weaker than linear.

Acknowledgments

This work was supported by Grants-in-Aid for Scientific Research (A) (No. 24244053) and (C) (No. 26400355) from the Ministry of Education, Culture, Sports, Science and Technology of Japan. One of the authors (I. P.) acknowledges the Russian Foundation for Basic Research, Grant No. 14-02-92104.

Appendix A. Analytical expressions for $\sigma_{\parallel}(0, \mu)$

Here we discuss several analytical expressions for $\sigma_{\parallel}(0, \mu)$ in equation (15) which can be useful for numerical calculations. We suppose that self-energy in equation (12) depends only on μ . The real part of the self-energy is absorbed in the chemical potential for brevity. Let us start with the summation over η in the expression

$$\sum_{\eta} \left[\frac{\epsilon_z^2}{E_n^2} \mathcal{A}_n^{(\eta)^2} + \left(1 - \frac{\epsilon_z^2}{E_n^2} \right) \mathcal{A}_n^{(\eta)} \mathcal{A}_n^{(-\eta)} \right]. \quad (\text{A.1})$$

At first, we note that

$$\mathcal{A}_n^{(\eta)} \mathcal{A}_n^{(-\eta)} = \frac{i\Gamma}{4\mu} \left(\frac{1}{E_n^2 - \mu_-^2} - \frac{1}{E_n^2 - \mu_+^2} \right), \quad (\text{A.2})$$

where the shorthand notation $\mu_{\pm} = \mu \pm i\Gamma$ is introduced. Next, we regroup the terms in equation (A.1) and make the η summation in the expression

$$\sum_{\eta} \left(\mathcal{A}_n^{(\eta)^2} - \mathcal{A}_n^{(\eta)} \mathcal{A}_n^{(-\eta)} \right) = -E_n^2 \left(\frac{1}{E_n^2 - \mu_-^2} - \frac{1}{E_n^2 - \mu_+^2} \right)^2, \quad (\text{A.3})$$

where the following identity was used

$$\mathcal{A}_n^{(\eta)} - \mathcal{A}_n^{(-\eta)} = -i\eta E_n \left(\frac{1}{E_n^2 - \mu_-^2} - \frac{1}{E_n^2 - \mu_+^2} \right). \quad (\text{A.4})$$

Combining together equations (A.2) and (A.3), we arrive to the following representation of expression (A.1)

$$\sum_{\eta} \left[\frac{\epsilon_z^2}{E_n^2} \mathcal{A}_n^{(\eta)^2} + \left(1 - \frac{\epsilon_z^2}{E_n^2} \right) \mathcal{A}_n^{(\eta)} \mathcal{A}_n^{(-\eta)} \right] = \frac{2(\epsilon_z^2 + \Gamma^2)}{(E_n^2 - \mu_-^2)(E_n^2 - \mu_+^2)} - \frac{\epsilon_z^2}{(E_n^2 - \mu_-^2)^2} - \frac{\epsilon_z^2}{(E_n^2 - \mu_+^2)^2}. \quad (\text{A.5})$$

With the help of equation (A.5), the conductivity in equation (15) can be rewritten in the following form

$$\sigma_{\parallel}(0, \mu) = \frac{e^2 \epsilon_c^2}{4\pi^3 \hbar^2 v} \int_0^{\infty} d\epsilon \sum_{n=0}^{\infty} \alpha_n \left[\frac{2(\epsilon^2 + \Gamma^2)}{(\epsilon_c^2 n + \epsilon^2 + \Delta_+^2)(\epsilon_c^2 n + \epsilon^2 + \Delta_-^2)} - \frac{\epsilon^2}{(\epsilon_c^2 n + \epsilon^2 + \Delta_+^2)^2} - \frac{\epsilon^2}{(\epsilon_c^2 n + \epsilon^2 + \Delta_-^2)^2} \right]. \quad (\text{A.6})$$

where $\Delta_{\pm}^2 = \Delta^2 - \mu_{\pm}^2$. In equation (A.6) either summation over n or integration over ϵ can be performed analytically. In the following, we consider these two cases separately.

High magnetic fields. In the case of high magnetic fields, it is reasonable to make integration over ϵ since the summation over Landau levels can be usually restricted to several dozens of terms. The integration is performed straightforwardly

$$\sigma_{\parallel}(0, \mu) = \frac{e^2 \epsilon_c^2}{4v\hbar^2} \sum_{n=0}^{\infty} \alpha_n \frac{4\Gamma^2 - \left(\sqrt{\epsilon_c^2 n + \Delta_+^2} - \sqrt{\epsilon_c^2 n + \Delta_-^2} \right)^2}{\sqrt{(\epsilon_c^2 n + \Delta_+^2)(\epsilon_c^2 n + \Delta_-^2)} \left(\sqrt{\epsilon_c^2 n + \Delta_+^2} + \sqrt{\epsilon_c^2 n + \Delta_-^2} \right)}. \quad (\text{A.7})$$

Note that the series over n is convergent. In the important practical case $n = 0$ and $\Delta = 0$ this expression gives simple answer [8]

$$\sigma_0 = \frac{e^2}{2\hbar^2 v} \frac{\epsilon_c^2}{\Gamma}. \quad (\text{A.8})$$

Another important example is the conductivity at $\mu = 0$. In this case $\Delta_+^2 = \Delta_-^2 = \Delta^2 + \Gamma^2$, and equation (A.7) gives

$$\sigma_{\parallel}(0, 0) = \frac{e^2 \Gamma^2}{v\hbar^2 \epsilon_c} \left[\zeta \left(\frac{3}{2}, \frac{\Delta^2 + \Gamma^2}{\epsilon_c^2} \right) - \frac{1}{2} \frac{\epsilon_c^3}{(\Delta^2 + \Gamma^2)^{3/2}} \right]. \quad (\text{A.9})$$

In the $\epsilon \rightarrow 0$ limit, with the help of equation (22), this expression gives

$$\sigma_{\parallel}(0, 0) = \frac{2e^2}{v\hbar^2} \frac{\Gamma^2}{\sqrt{\Delta^2 + \Gamma^2}}. \quad (\text{A.10})$$

Moderate magnetic fields. In order to study a behaviour at low magnetic fields, it maybe useful to perform a summation in equation (A.6) via special functions. With the help of the definition of Hurwitz ζ -function and using the identity

$$\sum_{n=0}^{\infty} \frac{1}{(n+a)(n+b)} = \frac{1}{b-a} (\psi(b) - \psi(a)), \quad (\text{A.11})$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is digamma function, we obtain $\sigma_{\parallel} = \sigma_{\text{I}} + \sigma_{\text{II}}$ where

$$\sigma_{\text{I}} = \frac{e^2}{\pi^3 \hbar^2 v} \int_0^\infty d\epsilon \left\{ \frac{(\epsilon^2 + \Gamma^2)}{\Delta_+^2 - \Delta_-^2} \left[\psi\left(\frac{\epsilon^2 + \Delta_+^2}{\epsilon_c^2}\right) - \psi\left(\frac{\epsilon^2 + \Delta_-^2}{\epsilon_c^2}\right) + \frac{1}{2} \frac{\epsilon_c^2}{\epsilon^2 + \Delta_+^2} - \frac{1}{2} \frac{\epsilon_c^2}{\epsilon^2 + \Delta_-^2} \right] \right\}, \quad (\text{A.12})$$

and

$$\sigma_{\text{II}} = -\frac{e^2}{2\pi^3 \hbar^2 v \epsilon_c^2} \int_0^\infty d\epsilon \epsilon^2 \left[\zeta\left(2, \frac{\epsilon^2 + \Delta_+^2}{\epsilon_c^2}\right) + \zeta\left(2, \frac{\epsilon^2 + \Delta_-^2}{\epsilon_c^2}\right) - \frac{1}{2} \frac{\epsilon_c^4}{(\epsilon^2 + \Delta_+^2)^2} - \frac{1}{2} \frac{\epsilon_c^4}{(\epsilon^2 + \Delta_-^2)^2} \right]. \quad (\text{A.13})$$

Note that σ_{I} and σ_{II} are introduced for convenience and have no individual meaning since the integrals over ϵ are divergent. However, $\sigma_{\text{I}} + \sigma_{\text{II}}$ is convergent.

Zero magnetic field. The $\epsilon_c \rightarrow 0$ limit of σ_{I} can be studied using the expansion $\psi(z) = \log z - 1/(2z) - 1/(12z^2) + \dots$, which gives

$$\sigma_{\text{I}} = \frac{e^2}{\pi^3 \hbar^2 v} \int_0^\infty d\epsilon (\epsilon^2 + \Gamma^2) \frac{\log(\epsilon^2 + \Delta_+^2) - \log(\epsilon^2 + \Delta_-^2)}{\Delta_+^2 - \Delta_-^2}, \quad (\text{A.14})$$

where

$$\frac{\log(\epsilon^2 + \Delta_+^2) - \log(\epsilon^2 + \Delta_-^2)}{\Delta_+^2 - \Delta_-^2} = \frac{1}{2|\mu|\Gamma} \begin{cases} \tan^{-1} \frac{2|\mu|\Gamma}{\epsilon^2 + \Delta^2 + \Gamma^2 - \mu^2}, & \epsilon^2 > \mu^2 - \Delta^2 - \Gamma^2, \\ \pi + \tan^{-1} \frac{2|\mu|\Gamma}{\epsilon^2 + \Delta^2 + \Gamma^2 - \mu^2}, & \epsilon^2 < \mu^2 - \Delta^2 - \Gamma^2. \end{cases} \quad (\text{A.15})$$

Finally, we arrive to the following expression for σ_{I} :

$$\sigma_{\text{I}} = \frac{e^2}{\pi^3 \hbar^2 v} \int_0^\infty d\epsilon \frac{\epsilon^2 + \Gamma^2}{2\mu\Gamma} \tan^{-1} \frac{2\mu\Gamma}{\epsilon^2 + \Delta^2 + \Gamma^2 - \mu^2}, \quad \text{if } \mu < \sqrt{\Delta^2 + \Gamma^2}, \quad (\text{A.16})$$

and

$$\sigma_{\text{I}} = \frac{e^2}{\pi^3 \hbar^2 v} \int_0^{\sqrt{\mu^2 - \Delta^2 - \Gamma^2}} d\epsilon \frac{\epsilon^2 + \Gamma^2}{2|\mu|\Gamma} \left(\pi + \tan^{-1} \frac{2|\mu|\Gamma}{\epsilon^2 + \Delta^2 + \Gamma^2 - \mu^2} \right) + \frac{e^2}{\pi^3 \hbar^2 v} \int_{\sqrt{\mu^2 - \Delta^2 - \Gamma^2}}^\infty d\epsilon \frac{\epsilon^2 + \Gamma^2}{2\mu\Gamma} \tan^{-1} \frac{2\mu\Gamma}{\epsilon^2 + \Delta^2 + \Gamma^2 - \mu^2}, \quad \text{if } \mu > \sqrt{\Delta^2 + \Gamma^2}. \quad (\text{A.17})$$

The zero-field limit for σ_{II} is straightforward

$$\sigma_{\text{II}} = -\frac{e^2}{2\pi^3 \hbar^2 v} \int_0^\infty d\epsilon \epsilon^2 \left(\frac{1}{\epsilon^2 + \Delta_+^2} + \frac{1}{\epsilon^2 + \Delta_-^2} \right). \quad (\text{A.18})$$

As we mentioned above, although each of σ_{I} and σ_{II} diverges, the physical meaning has only the sum $\sigma_{\text{I}} + \sigma_{\text{II}}$ which is well defined, as it can be clearly seen in the $\mu = 0$ case where equations (A.16) and (A.18) give the same result as equation (A.10).

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